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ORIGINAL PAPER

Periodic solution of a turbidostat system with impulsive state feedback control

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Abstract A turbidostat is an apparatus used to continuously culturing microorganisms. The concentration of the microorganism in the turbidostat can be controlled by the photoelectricity system or other devices. When the concentration of the microorganism is lower than a critical level, the dilution rate keeps constant. Once the concentration reaches the critical level, the dilution rate can be increased by the control of the photoelectricity system. Based on the design ideas of the turbidostat, a differential equation with impulsive state feedback control, which has no explicit solutions, is proposed for the turbidostat system. By the existence criteria of periodic solution of a general planar impulsive autonomous system, the conditions for the existence of an order one periodic solution of the system are obtained. Furthermore, it is pointed out that the system either tends to a stable state or has a periodic solution. Finally, some discussions and numerical simulations are given.

Keywords Turbidostat · Microorganism · Continuous culture · Impulsive control · Periodic solution

1 Introduction

Bioreactor control has become an active area of research on the continuous culture of microorganism in recent years [1]. The chemostat is an important laboratory apparatus

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used to continuously culturing microorganisms (see, for example, [2–11] and the references therein) [8]. It can be used to investigate microbial growth and has the advantage that the parameters are easy to measure. However, with the growth of the microorganism and its concentration increasing in the chemostat, the effect of inhibition between the production and other negative effect will occur when the concentration of the microorganism reaches a critical value. For the purpose of continuously culturing the microorganism and decreasing the inhibition effect, it is necessary to keep the concentration of microorganism lower than a certain level. So the chemostat with the feedback control of the dilution rate, which is often referred to as a turbidostat by bio-engineers and biologists [12], is established. In this sense, the turbidostat is actually a continuous culture apparatus (see Fig. 1), similar to the chemostat, which has feedback between the turbidity of the culture vessel and the dilution rate (from http://en.wikipedia.org/wiki/Turbidostat). In the turbidostat, an optical sensor measures the turbidity of the fluid and this signal is used to control the dilution rate [12]. There are some papers investigating the mathematical models on the turbidostat, for example, coexistence of two species in the turbidostat was shown numerically by Flegr [13], and later analytically by De Leenheer and Smith [11]. Tang and Chen [14], Jiang et al. [15] and Smith [16] have studied the state-dependent models with impulsive state control, where the model has a first integral, and obtained the complete expression of the period of the periodic solution. Jiang et al. [17] and Zeng et al. [18] have also discussed the models concerning integrated pest management (IPM), which have no explicit solution, by applying the Poincare principle and Poincare-Bendixson of the impulsive differential equation, respectively. In addition, some papers (e.g. [9,19]) have investigated the impulsive differential system in the fields of biological mathematics. However few papers have discussed the system concerning the turbidostat using impulsive differential equation with the state feedback control.

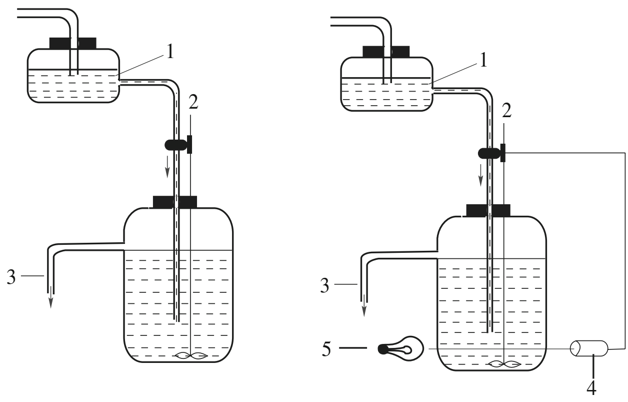

Fig. 1 The sketch map of chemostat and turbidostat. (1) Reservoir of sterile medium; (2) valve controlling flow of medium; (3) outlet for spent medium; (4) photo cell; (5) light source (the resource of turbidostat from: http://www.studentsguide.in/microbiology/microbial-nutrition-growth/chemostat-and-turbidostat.html)

In this paper, we will discuss the turbidostat model with the impulsive state feedback control according to the existence criteria of periodic solution of the general impulsive autonomous system in [18]. The remainder of the paper is organized as follows. In Sect. 2, we formulate a differential equation with the impulsive state feedback control according to the design ideas of the turbidostat. Some definitions, existence criteria of the periodic solution of a general planar impulsive autonomous system and related theorems are given. In Sect. 3, the qualitative analysis is given and furthermore the existences of order one periodic solution is investigated. Finally, some discussions and numerical simulations are provided in Sect. 4.

2 Model formulation and preliminaries

The general model of continuously culturing microorganism in a chemostat is given by the following form of differential equations [11]:

$$\begin{cases} \dfrac{dS}{dt} = Q(S^0 - S) - \dfrac{1}{\delta} f(S)x, \\ \dfrac{dx}{dt} = f(S)x - Qx, \end{cases}$$

where

(1) $S = S(t)$ is the concentration of substrate (or nutrient) and $x = x(t)$ is the concentration of microorganism in the chemostat at time t.
(2) Q is the *dilution rate* of the chemostat (or equivalently, $1/Q$ is the *residence time* of a molecule inside the chemostat) and S^0 is the concentration of the input substrate. The constant δ is the *yield constant*.
(3) The function f is called *uptake function* and satisfies the followings [11]:
- $(Regularity)$ $f : R^+ \to R^+$ is continuously differential and $f(0) = 0$;
- $(Monotonicity)$ f is monotonically increasing, i.e. $df/dS > 0$ for all $S \in R^+$.

Here, we take $f(S) = \mu S$ and then the above model has the following form:

$$\begin{cases} \dfrac{dS_1}{dt'} = Q(S_1^0 - S_1) - \dfrac{\mu}{\delta} S_1 x_1, \\ \dfrac{dx_1}{dt'} = \mu S_1 x_1 - Q x_1, \end{cases} \tag{2.1}$$

where $S_1 = S_1(t')$ and $x_1 = x_1(t')$ are the concentrate of substrate and microorganism at moment t', $\mu > 0$ is the consuming rate of x_1 to S_1. $\delta > 0$ is the yield rate. Here, $Q > 0$ is the dilution rate and S_1^0 is the initial concentration of the substrate which is input continuously.

According to the design ideas of the turbidostat, it is required that the concentration of the microorganism should be controlled to a certain level which is not larger than the critical value denoted by x_h. It is considered in this paper that the substrate with the microorganism is discharged and the additional substrate is input impulsively on the basis of the chemostat when x_1 reaches x_h. Therefore, (2.1) can be modified as follows by introducing the impulsive state feedback control:

$$\begin{cases} \left.\begin{aligned} \frac{dS_1}{dt'} &= Q(S_1^0 - S_1) - \frac{\mu}{\delta} S_1 x_1, \\ \frac{dx_1}{dt'} &= \mu S_1 x_1 - Q x_1, \end{aligned}\right\} x_1 < x_h, \\ \left.\begin{aligned} \Delta S_1 &= Q_1(S_1^0 - S_1), \\ \Delta x_1 &= -Q_1 x_1, \end{aligned}\right\} \quad x_1 = x_h, \\ S_1(0) = S_{10}, x_1(0) = x_{10} \end{cases} \tag{2.2}$$

where Q_1 is the impulsive dilution rate controlled by the photoelectricity system when the concentrate x_1 reaches x_h. For the convenience of discussion, let

$$S_1 = S_1^0 S, x_1 = \frac{\delta Q}{\mu} x, \ t' = \frac{1}{Q} t,$$

then (2.2) becomes

$$\begin{cases} \left.\begin{aligned} \frac{dS}{dt} &= 1 - S - Sx, \\ \frac{dx}{dt} &= aSx - x, \end{aligned}\right\} x < h, \\ \left.\begin{aligned} \Delta S &= b - bS, \\ \Delta x &= -bx, \end{aligned}\right\} \quad x = h, \\ S(0) = S_0, x(0) = x_0 \end{cases} \tag{2.3}$$

where

$$a = \frac{\mu S_1^0}{Q}, h = \frac{\mu x_h}{\delta Q}, \ b = Q_1.$$

In the following, we mainly discuss the existence of periodic solution of (2.3) by the existence criteria of periodic solution of the general impulsive autonomous system. Before introducing the existence criteria, we give the following definitions and lemmas [18]:

Definition 2.1 (*Lakshmikantham, et al. [20]*) An triple (X, π, R_+) is said to be a semi-dynamical system if X is a metric space, R_+ is the set of all non-negative reals and $\pi : X \times R_+ \to X$ is a continuous function such that

(i) $\pi(x, 0) = x$ for all $x \in X$;
(ii) $\pi(\pi(x, t), x) = \pi(x, t + s)$ for all $x \in X$ and $t, s \in R_+$.

we denote some times a semi-dynamical system (X, π, R_+) by (X, π).

For any $x \in X$, the function $\pi_x : R_+ \to X$ defined by $\pi_x(t) = \pi(x, t)$ is continuous and we call π_x the trajectory of x. The set $C^+(x) = \{\pi(x, t) | t \in R_+\}$ is called the positive orbit of x. For any subset M of X, we let $M^+(x) = C^+(x) \bigcap M - x$ and $M^-(x) =$

$G(x) \bigcap M - x$, where $G(x) = \bigcup\{G(x,t)|t \in R_+\}$ and $G(x) = \{y|\pi(y,t) = x\}$ is the attainable set of x at $t \in R_+$. Finally we set $M(x) = M^+(x) \bigcup M^-(x)$.

Definition 2.2 (*Lakshmikantham, et al. [20]*) An impulsive semi-dynamical system $(X, \pi; M, I)$ consists of a semi-dynamical system (X, π) together with a nonempty closed subset M of X and a continuous function $I : M \to X$ such that the following properties hold:

(i) No point $x \in X$ is a limit point of $M(x)$,
(ii) $[t|G(x,t) \bigcap M \neq \varnothing]$ is a closed subset of R^+.

According to the denotations in [18], we write $N = I(M) = \{y \in X|y = I(x), x \in M$ and for any $x \in X, I(x) = x^+\}$. Here in this paper, M is called the set of impulses, I is referred to the impulsive function.

Defining a function $\Phi : X \to R_+ \bigcup\{\infty\}$ as follows:

$$\Phi(x) = \begin{cases} \infty \text{ if } M^+(x) = \varnothing, \\ s \quad \text{if } \pi(x,t) \notin M \text{ for } 0 < t < s \text{ and } \pi(x,s) \in M, \end{cases}$$

Here s is called the time without impulse of x, i.e. s is the first time when $\pi(x, 0)$ hits M.

Definition 2.3 (*Lakshmikantham, et al. [20]*). Let $(X, \pi; M, I)$ be an impulsive semi-dynamical system and let $x \in X$ and $x \notin M$. The trajectory of x is a function $\widetilde{\pi}_x$ defined on subset $[0, s)$ of R_+ (s may be ∞) to X inductively as following:

$$\widetilde{\pi}_x(t) = \widetilde{\pi}(x^+_{n-1}, t), \quad \tau_{n-1} \le t < \tau_n,$$

where $\{x_n\}$ is the sequence of impulse points of x, which satisfied $\pi(x^+_{n-1}, \Phi(x^+_{n-1})) = x_n$. τ_n is the sequence of time of impulses relative to $\{x_n\}$, $\tau_n = \sum_{k=0}^{n-1} \Phi(x^+_k)$.

Definition 2.4 (*Lakshmikantham, et al. [20]*). A trajectory $\widetilde{\pi}_x$ is said to be periodic of period τ and order k if there exist positive integers $m \ge 1$ and $k \ge 1$ such that k is the smallest integer for which $x^+_m = x^+_{m+k}$ and $\tau = \sum_{i=m}^{m+k-1} \Phi(x^+_i)$.

Definition 2.5 (*Corless, et al. [21]*). Lambert$W$ function W is defined as multiple valued inverse function of $f : y \to ye^y = x$, we have $W(x)e^{W(x)} = x$ and its derivative satisfies:

$$x(1 + W(x))W'(x) = W(x),$$

when $x \neq 0$ and $x \neq -1/e$. $W(x)$ has two branches when $x \ge -1/e$, here we define $W(0, x)$ as principal branch satisfying $W(0, x) \ge -1$ and another branch as $W(-1, x)$ satisfying $W(-1, x) \le -1$. We can easily get the basic properties of function Lambert W:

$$\lim_{x\to 0} W(0, x) = 0, \lim_{x\to 0^-} W(-1, x) = -\infty.$$

More details can be found in [21].

Theorem 2.1 (Brouwers fixed-point theorem (Griffel [22])). *Every continuous mapping of a closed bounded convex set in R^n into itself has a fixed point.*

The existence criteria for an impulsive autonomous system with state-dependent has been proved by Theorem 2.1 (Brouwers fixed-point theorem) in [18]. For the convenience of reading, we repeat the main results of [18].

Consider the following general autonomous impulsive differential equations:

$$\begin{cases} \left.\begin{aligned} \frac{dx}{dt} &= P(x, y) \\ \frac{dy}{dt} &= Q(x, y) \end{aligned}\right\} (x, y) \notin M, \\ \left.\begin{aligned} \Delta x &= I_1(x, y) \\ \Delta y &= I_2(x, y) \end{aligned}\right\} (x, y) \in M. \end{cases} \tag{2.4}$$

Here $(x, y) \in R^2$, and P, Q, I_1, I_2 are all functions mapping R^2 into R, $M \subset R^2$ is the set of impulse, and we assume:

(H2.1) $P(x, y)$, $Q(x, y)$ are all continuous with respect to x, y in R^2.
(H2.2) $M \subset R^2$ is a line, $I_1(x, y)$ and $I_2(x, y)$ are linear functions of x and y.

For each point $S(x, y) \in M$, we define $I : R^2 \to R^2$:

$$I(S) = S^+ = (x^+, y^+) \in R^2, x^+ = x + I_1(x, y), y^+ = y + I_2(x, y).$$

Obviously $N = I(M)$ is also a line of R^2 or a subset of a line, and we assume that $N \bigcap M = \varnothing$. From Definition 2.2, we know (2.4) is an impulsive semi-dynamical system. The following theorem gives the conditions under which (2.4) has a periodic solution of order one defined by Definition 2.4.

Theorem 2.2 (Zeng et al. [18]) *If system (2.4) satisfies assumptions (H2.1) and (H2.2), and, there exists a bounded closed simply connected region D which has following properties:*

(i) There is no singularity in it and the boundary ∂D of D is composed of three parts: L_1, L_2 and L_3;
(ii) $L_1 = D \bigcap M$ cannot be tangent with trajectories of (2.4) except at end-points;
(iii) $L_2 \subset I(M)$ is a line segment which satisfied $I(L_1) \subset L_2$;
(iv) trajectories with initial point in $L_2 \bigcup L_3$ will enter into interior of D, then there must exist a periodic solution of system (2.4) of order one in region D.

3 Periodic solution of (2.3)

Before discussing the periodic solution of (2.3), we should consider the qualitative characteristics of (2.3) without the impulsive effect. If no impulsive effect is introduced, then (2.3) becomes

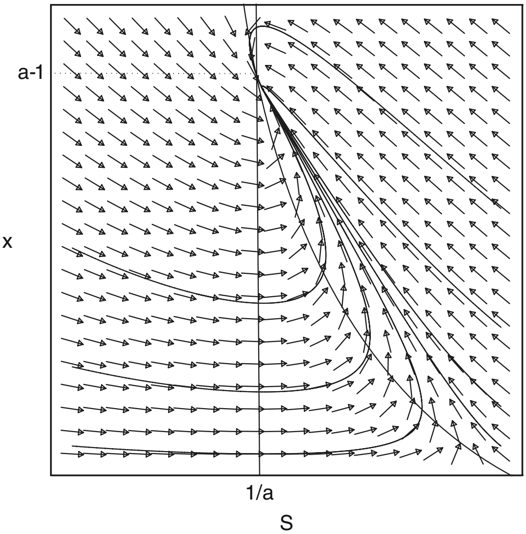

Fig. 2 Illustration of vector graph of (3.1) when $a > 1$

$$\begin{cases} \frac{dS}{dt} = 1 - S - Sx, \\ \frac{dx}{dt} = aSx - x, \end{cases} \tag{3.1}$$

The following results for (3.1) can be easily obtained and the proof is omitted.

(1) When $a \leq 1$, there exists a nonnegative equilibrium $(1, 0)$ which is stable. In this case, the microorganism is not cultured successfully.
(2) When $a > 1$, there exist two equilibria $(1, 0)$ and $\left(\frac{1}{a}, a - 1\right)$. $(1, 0)$ is a saddle point and $\left(\frac{1}{a}, a - 1\right)$ is globally asymptotically stable node. The vector graph of (3.1) can be seen in Fig. 2.

From the above discussions and Fig. 2, we know that $\left(\frac{1}{a}, a - 1\right)$ is a stable node when $a > 1$. For the initial point $x(0) < a - 1$ and $S(0) + S(0)x(0) \leq 1$, if $h > a - 1$, then all the solutions of (2.3) tend to the equilibrium $\left(\frac{1}{a}, a - 1\right)$ and no impulse will occur. If $h = a - 1$, the system tends to stable, it is unnecessary to control. So we mainly focus our attentions on the case $h < a - 1$, $x(0) < a - 1$ and $S(0) < 1$.

3.1 Existence of order one periodic solution

In order to apply the existence criteria (Theorem 2.2) of [18], we first need construct a closed region. our ideas to prove the existence of the periodic solution is to construct a closed region G_1 such that all the solutions of (2.3) enter the closed region and retain there. The ideas will be illustrated as follows using Fig. 3.

In Fig. 3, the line $x = h$ interacts the isoclinal line $\frac{dS}{dt} = 0$ at the point $A(S_A, h)$, $S_A = \frac{1}{1+h}$ and interacts the line $S = 1$ at the point $B(1, h)$. The impulsive set M

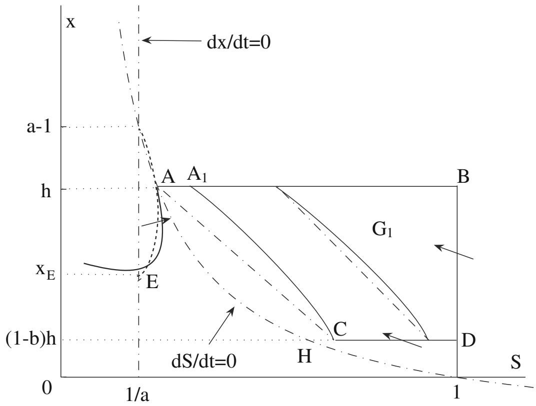

Fig. 3 Illustration of (2.3) for $a > 1, h < a - 1$

lies on the segment $\overline{AB}$, that is, $M \subseteq \overline{AB}$, $\overline{AB} = \{(S, x)|x = h, \frac{1}{1+h} \leq S \leq 1\}$. The impulsive function I_1 and I_2 map the impulsive set M as $N = I(M) \subseteq \overline{CD}$, $\overline{CD} = \{(S, x)|x = (1-b)h, b + \frac{1-b}{1+h} \leq S \leq 1\}$, where $C = C(S_C, (1-b)h)$, $D = D(1, (1-b)h)$, $S_C = b + \frac{1-b}{1+h}$.

Denote the arbitrary solution of (2.3) by (S, x). Suppose that the trajectory of (2.3) interacts the segment $\overline{AB}$. From the third equation of (2.3), we have $S^+ = b + (1-b)S$ for $x = h$ and further $S_i^+ = b + (1-b)S_i$, $i = 1, 2, \cdots$ when $x = h$. Since $S_i \geq S_A = \frac{1}{1+h}$, then $S_i^+ \geq b + \frac{1-b}{1+h}$. On the other hand, the line $x = (1-b)h$ interacts the curve $\frac{dS}{dt} = 0$ at the point $H(S_H, (1-b)h)$, $S_H = \frac{1}{1+(1-b)h}$ and the following inequality holds:

$$S_i^+ - \frac{1}{1+(1-b)h} \geq b + \frac{1-b}{1+h} - \frac{1}{1+(1-b)h} = \frac{bh(1-b)h}{(1+h)(1+(1-b)h)} \geq 0,$$

where $0 < b < 1$ (since b is the dilution rate). Therefore, after the first impulse, $S_i^+ \geq \frac{1}{1+(1-b)h}$. Suppose that the point H overlaps the point C, that is, $S_H = S_C$ and $b + \frac{1-b}{1+h} = \frac{1}{1+(1-b)h}$, then it follows $b = 1$, which is not suitable in the practice of production. Thus, $S_C > S_H$ and $S_i^+ > \frac{1}{1+(1-b)h}$.

Under the impulsive effect, the trajectory of (2.3) passing through the point A jumps to the point C. Since the point C lies in the region $\left\{(S, x)|\frac{dS}{dt} < 0, \frac{dx}{dt} > 0\right\}$, then we denote the interaction point of $\overline{AB}$ and the trajectory starting from C by $A_1(S_{A_1}, h)$. If $S_{A_1} = S_A$, then a periodic solution has been found. If $S_{A_1} > S_A$, then we can take the trajectory $\overset{\frown}{A_1C}$ as one boundary of the close region G_1. Other boundaries of G_1 are $\overline{CD}$, $\overline{DB}$ and $\overline{BA}$. Since

$$\frac{dx}{dt}|_{\overline{CD}} > 0, \frac{dS}{dt}|_{\overline{BD}} < 0,$$

then all the trajectories entering the closed region G_1 retain there and do not run out of the region.

During the above discussions, we suppose that the trajectories interact with the impulsive set M ($x = h$ or the segment $\overline{AB}$). But for the initial point $x(0) < a - 1$, $S(0) < 1$, there still exist some trajectories which do not interact $\overline{AB}$. In the following, we will give the sufficient condition under which the trajectory must interact with $\overline{AB}$.

Obviously, there exists one and only one trajectory of (2.3) which passes through the point A. Next, we introduce a comparison system

$$\begin{cases} \dfrac{dS}{dt} = S(h - x) \\ \dfrac{dx}{dt} = x(aS - 1) \end{cases} \tag{3.2}$$

System (3.2) has one positive equilibrium $\left(\frac{1}{a}, h\right)$ and all of its solutions are closed trajectories which satisfy:

$$V(S, x) \triangleq x + aS - h \ln x - \ln S + V_1^0 = C_1,$$

where $V_1^0 = h \ln h + \ln \frac{1}{a} - h - 1$, C_1 is an arbitrary constant. The derivative of $V(S, x)$ along the trajectories of (2.3) is

$$\frac{dV}{dt}|_{(2.3)} = \frac{1}{S}(aS - 1)(1 - S - hS).$$

When $\frac{1}{a} \le S \le \frac{1}{1+h}$, then $\frac{dV}{dt}|_{(2.3)} \ge 0$ which implies that the trajectories of (2.3) interacting with the trajectories of (3.2) passes through the trajectories of (3.2) from the left to the right(see Fig. 3). In addition, the trajectory of (3.2) passing through A interacts the line $S = \frac{1}{a}$ at the point $E\left(\frac{1}{a}, x_E\right)$ and the following equations holds:

$$h + \frac{a}{1 + h} - h \ln h - \ln \frac{1}{1 + h} = x_E + 1 - h \ln x_E - \ln \frac{1}{a}.$$

It follows that

$$x_E - h \ln x_E = C_2,$$

where

$$C_2 = h + \frac{a}{1 + h} - h \ln h - \ln \frac{1}{1 + h} - 1 + \ln \frac{1}{a}$$

and

$$x_E = -hW\left(0, -\frac{1}{h} \exp\left(\frac{-C_2}{h}\right)\right).$$

It can be easily shown that $-\frac{1}{h} \exp\left(\frac{-C_2}{h}\right) > -\frac{1}{e}$. Therefore, when $(1 - b)h < x_E$ for $S(0) + S(0)x(0) < 1$ and $x(0) < a - 1$ (see Fig. 3), the trajectories must interact $\overline{AB}$.

Furthermore, by the existence criteria of periodic solution in Theorem 2.2, we have the following theorem:

Theorem 3.1 *Suppose that $a > 1, h < a - 1$ and one of the following conditions holds:*

(1) $x(0) < h$ *and* $S(0) + S(0)x(0) \geq 1$*;*
(2) $(1 - b)h \leq x_E$, $x(0) \leq x_E$ *and* $S(0) + S(0)x(0) < 1$,

then system (2.3) has an order one periodic solution.

3.2 Existence of order $k(k \geq 2)$ periodic solution

From Theorem 3.1, we know that (2.3) has an order one periodic solution. In this subsection, we will discuss the stability of the periodic solution.

Suppose that $(\widetilde{S}, \widetilde{x})$ is the periodic solution of (2.3), then $(\widetilde{S}_0, \widetilde{x}_0) \in N \subseteq \overline{CD}$ and $(\widetilde{S}_1, \widetilde{x}_1) \in M \subseteq \overline{AB}$. The arbitrary trajectory (S, x) of (2.3) from the initial point $(S(0), x(0))$ interacts the set M $(x = h)$ at the point (S_1, h), by the effect of the impulse, the trajectory jumps to $(S_1^+, (1 - b)h)$ from (S_1, h). Consequently, the interaction points of the trajectory and the set M are $(S_2, h), (S_3, h), \ldots$, respectively. Under the effect of impulsive function I, the corresponding initial points after every impulse are $(S_1^+, (1 - b)h), (S_2^+, (1 - b)h), (S_3^+, (1 - b)h), \ldots$.

By the quality of the autonomic system, there is one and only one of the following sequences:

Case (a) $S_1 \leq S_2 \leq S_3 \leq \cdots$
and
Case (b) $S_1 \geq S_2 \geq S_3 \geq \cdots$.

We can assume that $\lim_{n\to\infty} S_n = \widetilde{S}_1$ (or $\lim_{n\to\infty} S_n^+ = \widetilde{S}_0$). Otherwise, suppose that above monotone sequences have no limitation in the region G_1, then it is implied that (2.3) has no periodic solution, which contradicts the conclusion of Theorem 3.1.

In addition, considering the monotony of the sequences, we know that the limitation is unique, furthermore, $\lim_{n\to\infty} S_n = \widetilde{S}_1$, that is, $S_1 \leq S_2 \leq S_3 \leq \cdots \leq \widetilde{S}_1$ or $S_1 \geq S_2 \geq S_3 \geq \cdots \geq \widetilde{S}_1$. By the proof of Proposition 3.3 in [15], it can also be obtained that there is no order $k(k \geq 2)$ periodic solution in system (2.3) and thus the system is not chaotic.

4 Discussions and numerical simulations

In this paper, we have investigated the existence of the periodic solution of a mathematical model concerning a turbidostat by the Poincare-Bendixson theorem and the existence criteria of periodic solution of a general impulsive autonomous system. Furthermore, the results show that (2.3) either tends to the stable state or has a periodic solution.

In order to verify the theoretical results in this paper, we next give the numerical simulations of (2.3). Let $a = 2, b = 0.1, S(0) = 0.4$, Fig. 4 gives the simulated

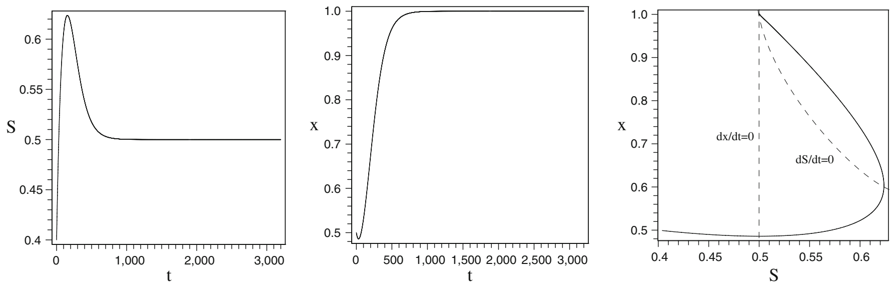

Fig. 4 The time series and portrait phase of (2.3) when $a = 2, b = 0.1, S(0) = 0.4, x(0) = 0.5$ and $h = 1.2$

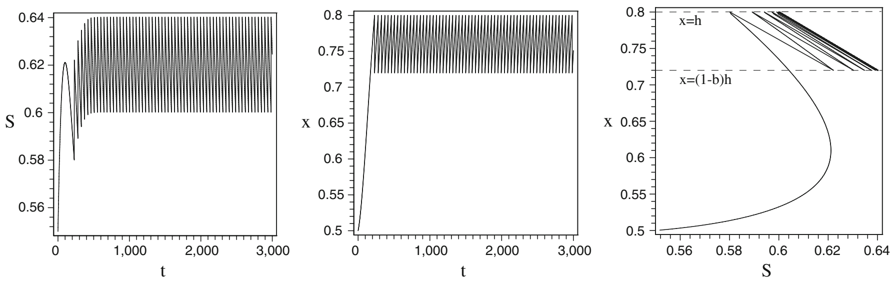

Fig. 5 The time series and portrait phase of (2.3) when $h = 0.8$, $S(0) = 0.55, x(0) = 0.5$ and other parameters are the same as those of Fig. 4

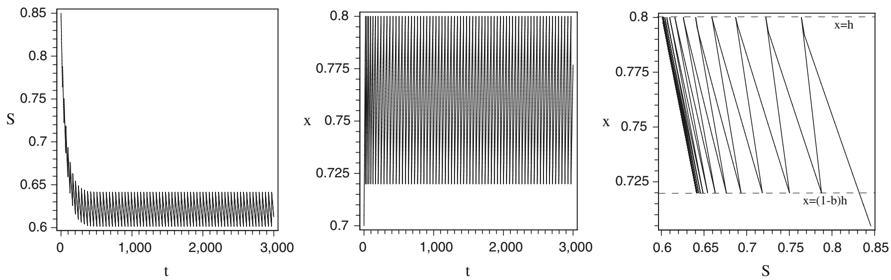

Fig. 6 The time series and portrait phase of (2.3) when $S(0) = 0.85, x(0) = 0.7$ and other parameters are the same as those of Fig. 5

time series and phase portrait when $x(0) = 0.5$ and $h = 1.2 > a - 1 = 1$. From Fig. 4, we know that no impulse occurs when $h > a - 1$. If we take $h = 0.8$ with $S(0) = 0.55, x(0) = 0.5$, the time series and phase portrait can be seen in Fig. 5. From Fig. 5, it can been seen that the trajectory tends to a periodic trajectory from the left side. Figure 6 ($S(0) = 0.85, x(0) = 0.7$ and $h = 0.8$) is the time series and the phase portrait and shows that the trajectory tends to a periodic trajectory from the right side.

A potential application area of a turbidostat with the feedback control is the commercial and industrial production of the microorganism. The microorganism in the turbidostat always keeps the highest growth rate and the concentration of the microorganism can be controlled to a given level. In the production practice, to obtain large number of cell or some of secondary metabolites, such as lactic acid, ethanol, one can use the turbidostat. In the paper, we have shown that the system either tends to the stable state (the equilibrium if $h \geq a - 1$) or has one periodic solution (if $h < a - 1$). According to the theoretical results, either the production of the microorganism tends to be stable or the production will be periodic. The key to the production by applying the turbidostat is to give the suitable feedback state (the value of x_h or h) and the control parameter (Q_1 or b). Firstly, the dilution rate Q_1 should be fixed according to the concentration S_1^0 of the substrate such that $a > 1$. Otherwise, if $a \leq 1$, then the microorganism could not be cultured successfully. Secondly, the impulsive dilution rate should be given according to the feedback state and the practice of the production. In addition, the initial states of the microorganism and the substrate should be considered in practice.

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